# §4.2 换元积分法

一、第一类换元法

二、第二类换元法

# 二、第二类换元法

❖定理2

设$x=\varphi(t)$是单调的、可导的函数，并且$\varphi'(t)\neq 0$.

又设$f[\varphi(t)]\varphi'(t)$具有原函数$F(t)$，则有换元公式

$$\int f(x)dx=\int f[\varphi(t)]\varphi'(t)dt$$

$$=F(t)+C=F[\varphi^{-1}(x)]+C.$$

其中$t=\varphi^{-1}(x)$是$x=\varphi(t)$的反函数.

这是因为，由复合函数和反函数求导法则，

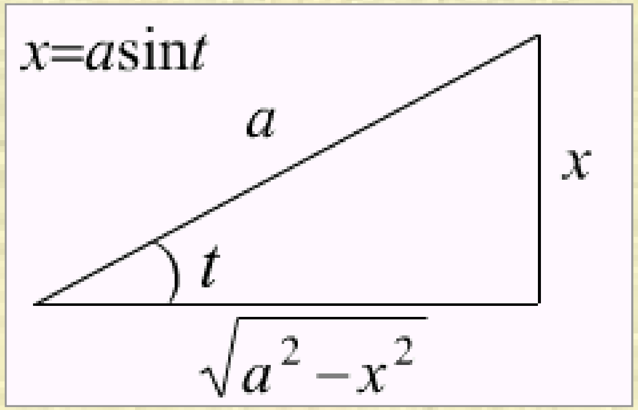
$$\{F[\varphi^{-1}(x)]\}'=F'(t)\frac{dt}{dx}=f[\varphi(t)]\varphi'(t)\frac{1}{\frac{dx}{dt}}=f[\varphi(t)]=f(x).$$

## ❖三角代换

•去根式 $\sqrt{a^2-x^2}$

作代换 $x=a\sin t,\quad t=\arcsin\dfrac{x}{a},$

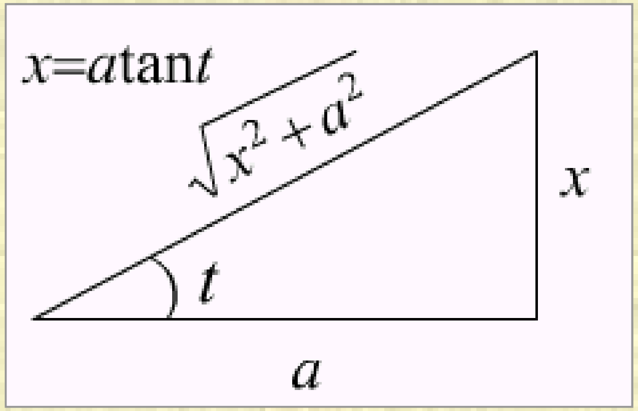
$$\sqrt{a^2-x^2}=a\cos t.$$




•去根式 $\sqrt{a^2+x^2}$

作代换 $x=a\tan t,\quad t=\arctan\dfrac{x}{a},$

$$\sqrt{a^2+x^2}=a\sec t.$$




•去根式 $\sqrt{x^2-a^2}$

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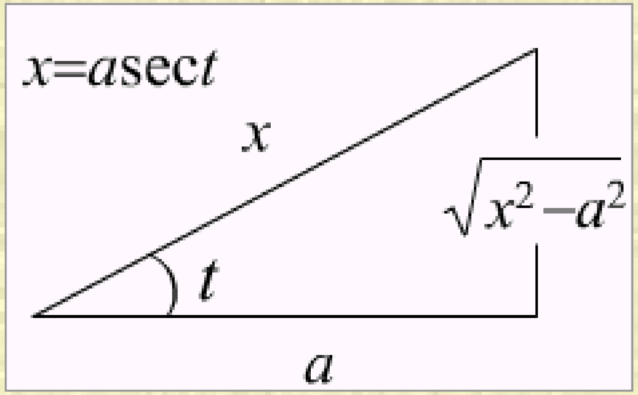


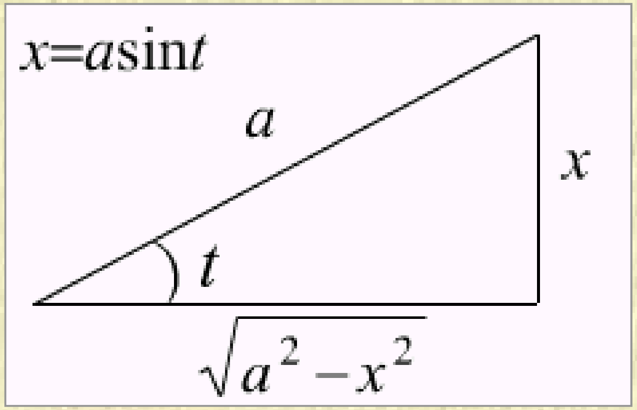
例1 求 $\int\sqrt{a^2-x^2}\,\mathrm{d}x \quad (a>0)$

解 令 $x=a\sin t \Rightarrow \mathrm{d}x=a\cos t\mathrm{d}t \quad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\int\sqrt{a^2-x^2}\,\mathrm{d}x=\int\sqrt{a^2-a^2\sin^2 t}\cdot a\cos t\mathrm{d}t$$

$$=\int a^2\cos^2 t\mathrm{d}t=a^2\int\frac{1+\cos 2t}{2}\mathrm{d}t$$

辅助三角形




$$=\frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)+C$$

$$=\frac{a^2}{2}(t+\sin t\cos t)+C$$

回代

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

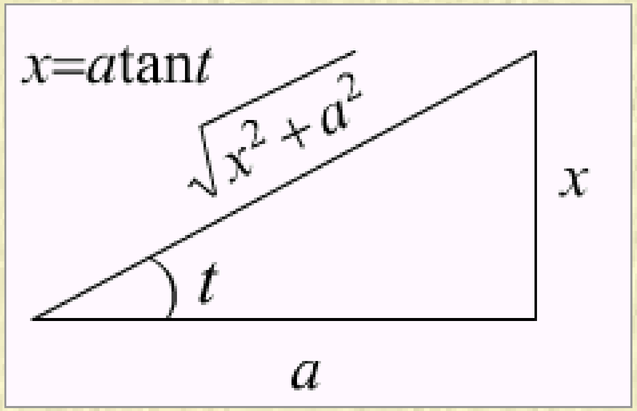
**例2** 求 $\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x \quad (a>0)$

**解** 令 $x = a\tan t \Rightarrow \mathrm{d}x = a\sec^2 t\mathrm{d}t \quad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \int \frac{1}{a\sec t}\cdot a\sec^2 t\mathrm{d}t$$

$$= \int \sec t\mathrm{d}t = \ln|\sec t + \tan t| + C_1$$

辅助三角形




回代

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right| + C_1$$

$$= \ln|x+\sqrt{x^2+a^2}| + C_1 - \ln a$$

$$= \ln|x+\sqrt{x^2+a^2}| + C$$

**例3** 求 $\int \frac{dx}{\sqrt{x^2-a^2}}\ (a>0)$.

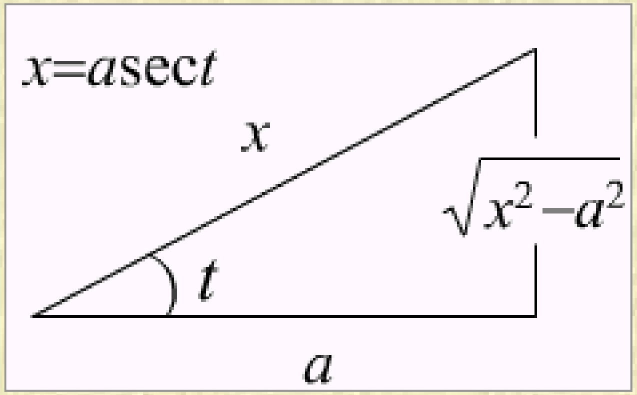
**解** 当 $x>a$ 时,

$$\int \frac{dx}{\sqrt{x^2-a^2}} \xlongequal{令x=a\sec t} \int \frac{a\sec t\tan t}{a\tan t}dt = \int \sec t dt = \ln|\sec t + \tan t| + C$$

回代

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C = \ln|x + \sqrt{x^2-a^2}| + C_1 \quad (C_1 = C - \ln a).$$

辅助三角形



例3 求 $\int \frac{dx}{\sqrt{x^2-a^2}}\ (a>0)$.

解 当$x>a$ 时,

$$\int \frac{dx}{\sqrt{x^2-a^2}} \overset{令x=a\sec t}{=\!=\!=} \int \frac{a\sec t\tan t}{a\tan t}dt=\int \sec t dt=\ln|\sec t+\tan t|+C$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C=\ln|x+\sqrt{x^2-a^2}|+C_1\quad (C_1=C-\ln a).$$

当$x<-a$时,

$$\int \frac{dx}{\sqrt{x^2-a^2}} \overset{令x=-u}{=\!=\!=} -\int \frac{du}{\sqrt{u^2-a^2}}=-\ln|u+\sqrt{u^2-a^2}|+C$$

$$=-\ln|-x+\sqrt{x^2-a^2}|+C=\ln|x+\sqrt{x^2-a^2}|+C_1\quad (C_1=C-2\ln a).$$

综合起来有 $\int \frac{dx}{\sqrt{x^2-a^2}}=\ln|x+\sqrt{x^2-a^2}|+C$ .

## ❖根式代换(去根式)

**例4** 求 $\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\mathrm{d}x$

**解** 令 $x = t^6$, $\mathrm{d}x = 6t^5\mathrm{d}t$

$$\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\mathrm{d}x = \int \frac{6t^5}{t^3(1+t^2)}\mathrm{d}t$$

$$= \int \frac{6t^2}{1+t^2}\mathrm{d}t = 6\int \frac{t^2+1-1}{1+t^2}\mathrm{d}t$$

$$= 6\int\left(1-\frac{1}{1+t^2}\right)\mathrm{d}t = 6[t-\arctan t]+C$$

$$= 6[\sqrt[6]{x}-\arctan\sqrt[6]{x}]+C$$

## ❖根式代换(去根式)

例5 求 $\int \frac{1}{\sqrt{1+e^x}}\mathrm{d}x$

解 令 $t=\sqrt{1+e^x}$, $e^x=t^2-1$,

$$x=\ln(t^2-1),\quad \mathrm{d}x=\frac{2t}{t^2-1}\mathrm{d}t.$$

$$\int \frac{1}{\sqrt{1+e^x}}\mathrm{d}x=\int \frac{1}{t}\cdot\frac{2t}{t^2-1}\mathrm{d}t$$

$$=\int \frac{2}{t^2-1}\mathrm{d}t=\ln\left|\frac{t-1}{t+1}\right|+C$$

$$=2\ln(\sqrt{1+e^x}-1)-x+C$$

注 一些情况下(如被积函数是分式, 分母的方幂较高时), 可作倒代换 $x=\frac{1}{t}$.

例6 求 $\int \frac{1}{x(x^7+2)}\mathrm{d}x$

解 令 $x=\frac{1}{t}$, $\mathrm{d}x=-\frac{1}{t^2}\mathrm{d}t$

$$\int \frac{1}{x(x^7+2)}\mathrm{d}x=\int \frac{t}{\left(\frac{1}{t}\right)^7+2}\cdot\left(-\frac{1}{t^2}\right)\mathrm{d}t=-\int \frac{t^6}{1+2t^7}\mathrm{d}t$$

$$=-\frac{1}{14}\int \frac{\mathrm{d}(1+2t^7)}{1+2t^7}=-\frac{1}{14}\ln|1+2t^7|+C$$

$$=-\frac{1}{14}\ln|2+x^7|+\frac{1}{2}\ln|x|+C$$

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注 一些情况下(如被积函数是分式, 分母的方幂较高时), 可作倒代换 $x=\frac{1}{t}$.

例7 求 $\int \frac{1}{x^3\sqrt{x^4+1}}\mathrm{d}x.$

解 令 $x=\frac{1}{t}$, $\mathrm{d}x=-\frac{1}{t^2}\mathrm{d}t$

$$\int \frac{1}{x^3\sqrt{x^4+1}}\mathrm{d}x=\int \frac{1}{t^{-3}\sqrt{t^{-4}+1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t$$

$$=-\int \frac{t^3}{\sqrt{1+t^4}}\mathrm{d}t=-\frac{1}{4}\int \frac{1}{\sqrt{1+t^4}}\mathrm{d}(t^4+1)$$

$$=-\frac{1}{2}\sqrt{1+t^4}+C=-\frac{\sqrt{x^4+1}}{2x^2}+C.$$

## ❖补充积分公式(P203)

$$\int \tan x dx = -\ln|\cos x| + C,$$

$$\int \cot x dx = \ln|\sin x| + C,$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C,$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C,$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\arctan\frac{x}{a} + C,$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C,$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\frac{x}{a} + C,$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C,$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + C.$$

例8 求$\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$

解 $\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$

$$=\int \frac{\mathrm{d}x}{\sqrt{(2x)^2+3^2}}$$

$$=\frac{1}{2}\int \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}}$$

$$=\frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right|+C$$

例9 求 $\int \frac{x\mathrm{d}x}{\sqrt{2x-x^2}}$

解 $\int \frac{x\mathrm{d}x}{\sqrt{2x-x^2}}$

$$= \int \frac{(x-1)\mathrm{d}x}{\sqrt{2x-x^2}} + \int \frac{\mathrm{d}x}{\sqrt{2x-x^2}}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}(2x-x^2)}{\sqrt{2x-x^2}} + \int \frac{\mathrm{d}(x-1)}{\sqrt{1-(x-1)^2}}$$

$$= -\sqrt{2x-x^2} + \arcsin(x-1) + C$$

练习

# 作 业

习题4–2 (P204):

2. (34) (35) (36) (37) (38) (39) (40)

例6 求$\int \frac{1}{x(x^7+2)}\mathrm{d}x$

解2 $\int \frac{1}{x(x^7+2)}\mathrm{d}x = \frac{1}{2}\int \frac{2+x^7-x^7}{x(x^7+2)}\mathrm{d}x$

$$= \frac{1}{2}\int \frac{1}{x}\mathrm{d}x - \frac{1}{2}\int \frac{x^6}{x^7+2}\mathrm{d}x$$

$$= \frac{1}{2}\ln|x| - \frac{1}{14}\int \frac{1}{x^7+2}\mathrm{d}(x^7+2)$$

$$= \frac{1}{2}\ln|x| - \frac{1}{14}\ln|2+x^7| + C$$

## ❖三角代换

练习1 求 $\int \frac{\sqrt{a^2-x^2}}{x^2}\mathrm{d}x = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{a} + C$

练习2 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+1}} = -\frac{\sqrt{x^2+1}}{x} + C$

练习3 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

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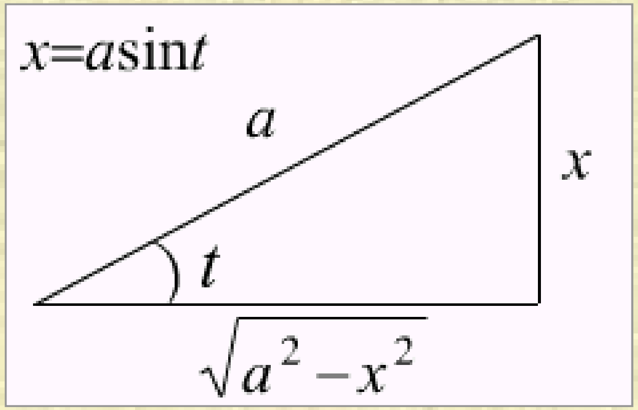


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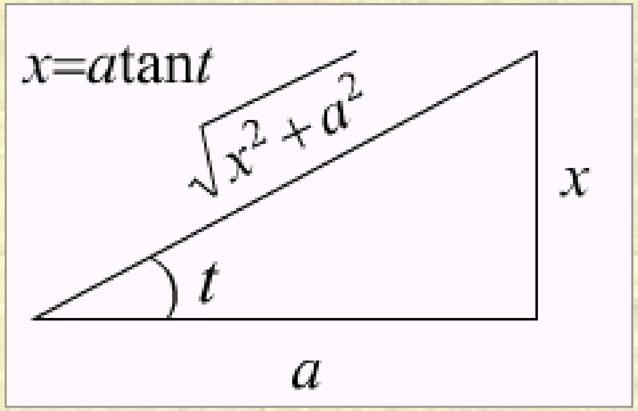


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$$\sqrt{x^2-a^2}=a\tan t.$$